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Moments of Class Numbers, III

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By means of the large sieve it is proved that

$$\sum_{n\leqslant x} h^{\alpha}(-n) = \vartheta(\alpha)\, x^{(\alpha+2)/2} + O(x^{(\alpha+2)/2-(1/4)+\epsilon})$$

holds for every $\alpha > 0$ and $\epsilon > 0$ ($h(-n)$ is the number of classes of primitive quadratic forms with discriminant $-n$).

Some applications of the large sieve to certain multiplicative functions are announced.

1. INTRODUCTION

In papers [7] and [8] I gave two slightly different methods of estimating the error term in the asymptotic formula for the sum $\sum_{n\leqslant x} h^k(-n)$ ($k = 1, 2,\ldots$). (For further references, see paper [7].) In [8] the error term was proved to be $\ll x^{[(k+2)/2]-(8-\epsilon)/k}$, if $\epsilon > 0$ and $k \geqslant k_0(\epsilon)$.

It is the aim of the present paper to improve this result and to extend it to powers not necessarily integral.

Write, as usual,

$$L(s, n) = \sum_{m\geqslant 1, 2\nmid m} \left(\frac{-n}{m}\right) m^{-s} \tag{1.1}$$

($\sigma = \operatorname{Re} s > 0$, $(-n/m)$ is the Jacobi symbol).

THEOREM 1. *For every real α and $\epsilon > 0$ we have*

$$\sum_{n\leqslant x} (L(1, n))^{\alpha} = \eta(\alpha)x + O(x^{3/4+\epsilon}).$$

$\eta(\alpha)$ will be defined in Lemma 1. The constants implied by the symbols $O(\ )$ and $\ll$ may depend on α and ϵ.

By Gauss' class-number formula and partial summation one easily gets from Theorem 1,

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THEOREM 2. *For every $\alpha > 0$ and $\epsilon > 0$ we have*

$$\sum_{n \leqslant x} h^{\alpha}(-n) = \theta(\alpha)\, x^{(\alpha+2)/2} + O(x^{(\alpha+2)/2-(1/4)+\epsilon}).$$

It is highly probable that in Theorem 1 the error term is $\ll x^{0.5+\epsilon}$. The difference depends on the fact that in the large-sieve inequality used below we sum over all primitive characters to certain moduli, whereas for the purpose of Theorem 1 we need estimations for real characters only. The inequalities for real characters proved in Elliott [2] and Wolke [7] do not give anything better.

By ideas similar to those of this paper and of Gallagher [4] one can prove the following:

Let $f(n)$ be a multiplicative arithmetical function with the properties

(i) $0 \leqslant f(p^a) \leqslant C_1 a^{C_2}$ *(p prime, $a = 1, 2, \ldots$),*

(ii) $\sum_{p \leqslant x} |f(p) - \tau| \ll x(\ln x)^{-\beta}$ *for $\tau = \tau(f) > 5/2$ and every $\beta > 0$.*

Then, for $\epsilon > 0$, there exists a $\gamma = \gamma(\epsilon, f) > 0$, such that

$$\sum_{k \leqslant x^{0.5-\epsilon}} \max_{(k,l)=1} \max_{y \leqslant x} \Bigg| \sum_{\substack{n \leqslant y \\ n \equiv l \bmod k}} f(n) - \frac{C(f)}{\phi(k)} \times \prod_{p|k} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right)^{-1} y \ln^{\tau-1} y \Bigg| \ll x(\ln x)^{\tau-\gamma} \quad (1.2)$$

holds.

(1.2) implies

$$\sum_{n \leqslant x} f(n)\, d(n-a) = c(f, a)\, x \ln^{\tau} x (1 + o(1)). \quad (1.3)$$

($d(n)$ is the divisor function, a is a fixed integer $\neq 0$).

In the special case $f(n) = d_k(n)$ ($k = 3, 4, \ldots$) this was shown by Linnik [5]. A detailed proof of (1.2) and (1.3) will appear elsewhere.

2. SOME LEMMAS

In the following let α and $\epsilon > 0$ be fixed and x be sufficiently large, $l_1 = \ln x$, $l_2 = \ln l_1$. For $\sigma > 1$ we have

$$(L(s, n))^{\alpha} = \prod_{p>2} \left(1 - \left(\frac{-n}{p}\right) p^{-s}\right)^{-\alpha} = \sum_{m=1}^{\infty} \left(\frac{-n}{m}\right) f_{\alpha}(m)\, m^{-s}, \quad (2.1)$$

where

$$f_\alpha(m) \ll (d(m))^{C(\alpha)}. \tag{2.2}$$

By $\mathscr{C}$ we denote the line from $1 + l_1^{-1} - il_1^2$ to $1 + l_1^{-1} + il_1^2$.

LEMMA 1. *We have*

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \Gamma(s-1)\, L^\alpha(s, n)\, Z^{s-1}\, ds = \eta(\alpha)x + O(x^{3/4+\epsilon}),$$

where $Z = x^{3/2}$ *and* $\eta(\alpha) = \sum_{m=1}^{\infty} \phi(m)/m^3 f_\alpha(m^2)$ ($\phi(m)$ *is Euler's function*).

The proof is similar to that of the corresponding lemma in Barban [1].

LEMMA 2. *Let* a_n ($M < n \leqslant M + N$) *be any complex numbers. Then the inequality*

$$\sum_{d \leqslant Q} \mu^2(d) \Big| \sum_{\substack{M<n\leqslant M+N \\ 2\nmid n}} a_n \left(\frac{-d}{n}\right)\Big|^2 \ll (Q^2 + N) \sum_{M<n\leqslant M+N} |\, a_n\,|^2$$

holds.

This follows immediately from the well-known large sieve inequality for character sums (see Gallagher [3]) and the fact that $\chi(n) = (-d/n)$ is a primitive character modulo either d or $4d$, if d is square-free.

For $y \in (x^\epsilon, x]$, $d \in (y, 2y]$, $\mu^2(d) = 1$ (in what follows d always denotes squarefree integers) we write

$$w = \exp(y^{\epsilon/2(|\alpha|+1)}), \tag{2.3}$$

and, for $\sigma \geqslant 1$,

$$\begin{aligned} L^\alpha(s, d) &= \prod_{2<p\leqslant y} \times \prod_{y<p\leqslant w} \times \prod_{p>w} \left(1 - \left(\frac{-d}{p}\right) p^{-s}\right)^{-\alpha} \\ &= L_1(s, d)\, L_2(s, d)\, L_3(s, d), \text{ say.} \end{aligned} \tag{2.4}$$

LEMMA 3. *For* $s \in \mathscr{C}$ *or* $s = 1$ *we have*

$$L_3(s, d) = 1 + O(y^{-2}).$$

We give the proof in the case $s = 1$.

By the Siegel–Walfisz prime number theorem (see Prachar [6]), we get

$$\sum_{p>w}\left(\frac{-d}{p}\right)p^{-1} = \int_w^\infty \frac{dt}{t^2}\sum_{k=1}^{d}\left(\frac{-d}{k}\right)(\pi(t,d,k)-\pi(w,d,k))$$

$$= \int_w^\infty \frac{dt}{t^2}\sum_{k=1}^{d}\left(\frac{-d}{k}\right)\left(\frac{\operatorname{Li} t}{\phi(d)}-\frac{\operatorname{Li} w}{\phi(d)}+O\left(\frac{t}{\phi(d)\ln^B t}\right)\right)$$

$$\ll \int_w^\infty \frac{dt}{t\ln^B t} \ll y^{-2}, \qquad \text{if} \quad B \geqslant B_0(\epsilon,\alpha).$$

Hence

$$\log L_3(1,d) = \alpha\sum_{p>w}\left(\frac{-d}{p}\right)p^{-1} + O(y^{-2}) \ll y^{-2},$$

which is equivalent to the statement of the Lemma.

LEMMA 4. *For $s \in \mathscr{C}$ or $s = 1$ we have*

$$\sum_{d\in(y,2y]} |1 - L_2(s,d)\,L_3(s,d)| \ll y^{3/4+2\epsilon}.$$

Proof. Clearly

$$L_2(s,d) \ll \prod_{p\leqslant w}\left(1+\frac{1}{p}\right)^{|\alpha|} \ll \exp\left(|\alpha|\sum_{p\leqslant w}p^{-1}\right) \ll y^{\epsilon} \tag{2.5}$$

and

$$\log L_2(s,d) = \alpha\sum_{y<p\leqslant w}\left(\frac{-d}{p}\right)p^{-s} + O(y^{-1}) = \alpha F(s,d) + O(y^{-1}), \text{ say.} \tag{2.6}$$

In the case $|F(s,d)| \leqslant 1$, we have

$$L_2(s,d) = 1 + O(|F(s,d)| + y^{-1}).$$

Otherwise we use (2.5): $1 - L_2(s,d) \ll y^{\epsilon}|F(s,d)|$. Hence, in either case,

$$1 - L_2(s,d) \ll y^{\epsilon}|F(s,d)| + y^{-1}. \tag{2.7}$$

(2.7) and Lemma 3 imply

$$\sum_{d\in(y,2y]}|1 - L_2(s,d)\,L_3(s,d)| \ll y^{\epsilon}\sum_{d\in(y,2y]}|F(s,d)| + y^{3/4}. \tag{2.8}$$

We split up the interval $(y, w]$ into $\ll y^{\epsilon}$ intervals $I_\nu = (u_\nu, u_{\nu+1}]$ with $u_{\nu+1} \leqslant 2u_\nu$. Writing

$$\left(\sum_{p\in I_\nu}\left(\frac{-d}{p}\right)p^{-s}\right)^2 = \sum_{u_\nu^2<n\leqslant u_{\nu+1}^2}\left(\frac{-d}{n}\right)a_n n^{-s},$$

we get, with Hölder's inequality and Lemma 2,

$$\sum_{d\in(y,2y]} |F(s,d)| \ll y^{3/4} \sum_{\nu} \Big(\sum_{d\leqslant 2y} \Big| \sum_{u_\nu^2<n\leqslant u_{\nu+1}^2} \Big(\frac{-d}{n}\Big) a_n n^{-s} \Big|^2 \Big)^{1/4}$$

$$\ll y^{3/4} \sum_{\nu} ((y^2+u_\nu^2)\, u_\nu^{-2})^{1/4} \ll y^{3/4+\epsilon}. \tag{2.9}$$

The Lemma now follows from (2.8) and (2.9).

For

$$5/6 \leqslant \sigma \leqslant 1 - \frac{\epsilon}{6} \tag{2.10}$$

let $\mathscr{C}_\sigma$ denote the path which connects the four points $1 + l_1^{-1} - il_1^2$, $\sigma - il_1^2$, $\sigma + il_1^2$, and $1 + l_1^{-1} + il_1^2$ and let $\mathscr{C}_\sigma'$ be the line connecting the points $\sigma - il_1^2$ and $\sigma + il_1^2$. By $A_\sigma = A_\sigma(y)$ we denote the number of $d \in (y, 2y]$ such that

$$|L_1(s,d)| > y^\epsilon \tag{2.11}$$

holds for at least one $s \in \mathscr{C}_\sigma$.

Lemma 5. *We have the inequality*

$$A_\sigma \ll y^{6(1-\sigma)+\epsilon}.$$

Proof. Put $y' = (\log y)^{1/(1-\sigma)}$. Because of

$$\prod_{p\leqslant y'} \Big(1 - \Big(\frac{-d}{p}\Big) p^{-s}\Big)^{-\alpha} \ll y^{\epsilon/2} \qquad \text{for} \quad s \in \mathscr{C}_\sigma$$

it is sufficient to estimate the number of $d \in (y, 2y]$ with

$$|G(s,d)| = \Big| \sum_{y'<p\leqslant y} \Big(\frac{-d}{p}\Big) p^{-s} \Big| > \frac{\epsilon}{3} \ln y \qquad \text{for at least one} \quad s \in \mathscr{C}_\sigma. \tag{2.12}$$

For simplicity, we estimate only A_σ', the number of $d \in (y, 2y]$ such that (2.12) holds for at least one $s \in \mathscr{C}_\sigma'$. The contribution of the horizontal parts of $\mathscr{C}_\sigma$ may be treated in the same manner.

We split up the interval $(y', y]$ into $\ll l_1$ intervals $(z_\nu, z_{\nu+1}]$ with $z_{\nu+1} \leqslant z_\nu$. The interval $[-l_1^2, l_1^2]$ we dissect into $\ll l_1^3$ parts $[w_\mu, w_{\mu+1}]$ with $|w_\mu - w_{\mu+1}| \leqslant l_1^{-1}$. Write

$$T_\nu(s,d) = \sum_{z_\nu<p\leqslant z_{\nu+1}} \Big(\frac{-d}{p}\Big) p^{-s}. \tag{2.13}$$

For $\nu \ll l_1$, $\mu \ll l_1^3$, and $0 \leqslant \rho \leqslant l_1^4$ let $A'_{\nu,\mu,\rho} = A'_{\nu,\mu,\rho}(\sigma, y)$ be the number of $d \in (y, 2y]$ for which

$$| T_\nu^{(\rho)}(\sigma + iw_\mu, d)| \geqslant (\ln z_{\nu+1})^{\rho-2} \tag{2.14}$$

holds. By Taylor's formula one easily sees that $| G(s, d)| \ll 1$ is true for all $s \in \mathscr{C}_\sigma'$ if d is not counted in any $A'_{\nu\mu\rho}$. Hence, by (2.12),

$$A_\sigma' \leqslant \sum_{\nu,\mu,\rho} A'_{\nu,\mu,\rho}. \tag{2.15}$$

Let ν, μ and ρ be fixed and write $z = z_{\nu+1}$. Clearly

$$A'_{\nu,\mu,\rho} \leqslant ((\ln z)^{\rho-2})^{-2r} \sum_{d \leqslant 2y} | T_\nu^{(\rho)}(\sigma + iw_\mu, d)|^{2r},$$

where

$$r = [2 \ln y/\ln z] + 1. \tag{2.16}$$

Obviously,

$$(T_\nu^{(\rho)}(s, d))^r = \sum_{z_\nu{}^r < n \leqslant z_{\nu+1}^r} \left(\frac{-d}{n}\right) n^{-s} a_n$$

with

$$a_n = \sum_{\substack{n=p_1\cdots p_r \\ p_j \in (z_\nu, z_{\nu+1}] \\ j=1,\ldots,r}} \ln^\rho p_1 \cdots \ln^\rho p_r \ll C^r r!(\ln z)^{\rho r}$$

and

$$\sum_n a_n \leqslant \Big(\sum_{z_\nu < p \leqslant z_{\nu+1}} \ln^\rho p\Big)^r \ll (\ln z)^{\rho r} z^r.$$

The last lines and Lemma 2 imply

$$A'_{\nu,\mu,\rho} \ll C^r r!(\ln z)^{4r} z^{2r(1-\sigma)}. \tag{2.17}$$

In the case $z \in (y', y^\epsilon]$ one easily verifies the inequalities

$$C^r \ll y^{\epsilon/4}, \qquad r! \ll y^{2(1-\sigma)+\epsilon/4}, \qquad \text{and} \qquad z^{2r(1-\sigma)} \ll y^{4(1-\sigma)+\epsilon/4}.$$

In the case $z \in (y^\epsilon, y]$, we have

$$C^r r! \ll 1 \qquad \text{and} \qquad z^{2r(1-\sigma)} \leqslant z^{(4(\ln y/\ln z)+2)(1-\sigma)} \leqslant y^{6(1-\sigma)}.$$

Hence, in both cases,

$$A'_{\nu,\mu,\rho} \ll y^{6(1-\sigma)+(3/4)\epsilon}. \tag{2.18}$$

Using (2.15), we finally get the required result.

3. Proof of Theorem 1

Every n can be written uniquely in the form $n = dR^2Q^2$ with $\mu^2(d) = 1$, $p \mid R \Rightarrow p \mid d$, and $(Q, d) = 1$. Obviously

$$L^\alpha(s, n) = L^\alpha(s, d)\, F_Q(s), \tag{3.1}$$

where $F_Q(s)$ is regular and $\ll x^\epsilon$ for $\sigma \geqslant \frac{1}{2}$. Hence

$$\sum_{n \leqslant x} L^\alpha(s, n) = \sum_{d \leqslant x} L^\alpha(s, d)\, G_d(s), \tag{3.2}$$

where, with the above notation,

$$G_d(s) = \sum_{R^2Q^2 \leqslant x/d} F_Q(s) \ll x^{1/2+\epsilon} d^{-1/2}. \tag{3.3}$$

For a fixed integer $\kappa \geqslant \epsilon^{-1}$ we write

$$\sigma_j = \frac{5}{6} + \frac{j}{6\kappa} \qquad (j = 0, 1, \ldots, \kappa - 1); \tag{3.4}$$

$\sum_\nu$ stands for summation over all intervals

$$J_\nu = (y_{\nu+1}, y_\nu] = (x2^{-\nu-1}, x2^{-\nu}]$$

with $\nu \geqslant 0$ and $x2^{-\nu} \geqslant x^\epsilon$.

For every ν we split up the set $\{d \mid d \in J_\nu\}$ into disjoint subsets $\mathscr{S}_{\nu,0}, \ldots, \mathscr{S}_{\nu,j-1}$, such that the following is valid:

$$| L_1(s, d)| \leqslant y_\nu^\epsilon \quad \text{ if } \; d \in \mathscr{S}_{\nu,j}, \; s \in \mathscr{C}\sigma_j, \text{ and } \; 0 \leqslant j \leqslant \kappa - 2. \tag{3.5}$$

By Lemma 5 one sees that the sets $\mathscr{S}_{\nu,j}$ may be chosen in such a way that

$$S_{\nu,j} = \operatorname{card}(S_{\nu,j}) \ll y_\nu^{6(1-\sigma j)+2\epsilon} \tag{3.6}$$

holds for $1 \leqslant j \leqslant \kappa - 1$. In particular,

$$S_{\nu,\kappa-1} \ll y_\nu^{3\epsilon} \qquad \text{for every } \nu. \tag{3.7}$$

We mention Siegel's result (see Prachar [6])

$$L^\alpha(s, n) \ll x^\epsilon, \qquad \text{if} \quad n \leqslant x \quad \text{and} \quad s \in \mathscr{C} \quad \text{or} \quad s = 1. \tag{3.8}$$

Because of (3.2) and (3.3) we can write, with $Z = x^{3/2}$,

$$\begin{aligned} I(x) &= \frac{1}{2\pi i} \sum_{n \leqslant x} \int_{\mathscr{C}} \Gamma(s-1)\, L^\alpha(s, n)\, Z^{s-1}\, ds \\ &= \frac{1}{2\pi i} \sum_\nu \int_{\mathscr{C}} \Gamma(s-1)\, Z^{s-1} \sum_{d \in J_\nu} L^\alpha(s, d)\, G_d(s)\, ds \\ &\quad + O\left(x^{0.5+2\epsilon} \sum_{d \leqslant x^\epsilon} \int_{\mathscr{C}} | \Gamma(s-1)\, L^\alpha(s, d)|\, |\, ds\, |\right), \end{aligned}$$

and further, by (2.4) and (3.8),

$$\begin{aligned} I(x) &= \frac{1}{2\pi i} \sum_\nu \int_{\mathscr{C}} \Gamma(s-1)\, Z^{s-1} \sum_{d \in J_\nu} L_1(s, d)\, G_d(s)\, ds \\ &\quad + O\left(x^{1/2+\epsilon} \sum_\nu y_\nu^{-1/2} \int_{\mathscr{C}} | \Gamma(s-1)| \right. \\ &\quad \left. \times \sum_{d \in J_\nu} | L_1(s, d)|\, |\, 1 - L_2 L_3(s, d)|\, |\, ds\, |\right) + O(x^{3/4}). \end{aligned} \tag{3.9}$$

By Lemma 4 and a trivial estimation for $L_1(s, d)$ the first error term is $\ll x^{1/2+4\epsilon} \sum_\nu y_\nu^{1/4} \ll x^{3/4+4\epsilon}$. For $d \in \mathscr{S}_{\nu,k-1}$ we estimate the integral trivially. Hence, by (3.7) and (3.8), the contribution of all sets $\mathscr{S}_{\nu,k-1}$ is $\ll x^{3/4}$.

For $d \in \mathscr{S}_{\nu,j}$ $(0 \leqslant j \leqslant \kappa - 2)$ we replace the integral over $\mathscr{C}$ by the integral over $\mathscr{C}_{\sigma_j}$. The pole of the Γ-function at the point $s = 0$ produces the residue $L_1(1, d)\, G_d(1)$. Collecting the last results, we get

$$\begin{aligned} I(x) &= \sum_\nu \sum_{0 \leqslant j \leqslant \kappa-2} \sum_{d \in \mathscr{S}_{\nu,j}} L_1(1, d)\, G_d(1) \\ &\quad + O\left(x^{1/2+\epsilon} \sum_\nu y_\nu^{-1/2} \sum_{0 \leqslant j \leqslant \kappa-2} Z^{\sigma_j - 1} \int_{\mathscr{C}_{\sigma_j}} \sum_{d \in \mathscr{S}_{\nu,j}} | L_1(s, d)|\, |\, ds\, |\right) \\ &\quad + O(x^{3/4+4\epsilon}). \end{aligned} \tag{3.10}$$

By calculations similar to those above one can show that the main term on the right side of (3.10) is

$$= \sum_{n \leqslant x} L^\alpha(1, n) + O(x^{3/4+4\epsilon}).$$

To the first error term we apply (3.5), (3.6), and (3.4):

$$I(x) - \sum_{n \leqslant x} L^{\alpha}(1, n) \ll x^{3/4+4\epsilon} + x^{1/2+4\epsilon} \sum_{\nu} y_{\nu}^{-1/2} \sum_{0 \leqslant j \leqslant \kappa-2} x^{3/2(\sigma_j-1)} y_{\nu}^{6(1-\sigma_j)}$$

$$\ll x^{3/4+4\epsilon} + x^{1/2+4\epsilon} \sum_{\nu} \sum_{0 \leqslant j \leqslant \kappa-2} y_{\nu}^{9/2(1-\sigma_j)-1/2}$$

$$\ll x^{3/4+4\epsilon} + x^{1/2+4\epsilon} \sum_{\nu} y_{\nu}^{1/4} \ll x^{3/4+5\epsilon}.$$

This, in connection with Lemma 1, proves Theorem 1.

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